Journal of Engineering Mathematics **23**: 141–155, 1989.

Similarity solutions for nonlinear diffusion – a new integration procedure

J.M. HILL
Department of Mathematics, University of Wollongong, Wollongong, N.S.W., Australia

Received 16 September 1988; accepted 1 November 1988

Abstract. The nonlinear diffusion equation arises in many important areas of science and technology and most of the known exact solutions turn out to be similarity solutions. For a general similarity solution involving an arbitrary parameter λ, a new integration procedure is proposed which enables first integrals to be obtained for special values of λ. The best known exact solutions arise from this analysis when the integration constant is taken to be zero and the procedure provides a natural way of deducing other special exact solutions. A new exact solution is obtained for the power law diffusivity of index $-4/3$ and new first integrals are deduced for a general equation which includes nonlinear cylindrical and spherical symmetrical diffusion and one-dimensional nonlinear diffusion with an inhomogeneous diffusivity. The procedure has given rise to an extensive number of first-order ordinary differential equations which include a wide variety of differing physical situations and which warrant further study either analytically to determine exact integrals or numerically for particular boundary value problems.

1. Introduction

The nonlinear diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(c)\frac{\partial c}{\partial x}\right), \tag{1.1}$$

arises in many areas in science and engineering and the literature on solutions, both exact and approximate, is scattered through many diverse disciplines. The soil science literature is cited by Crank [9] (page 105) and the majority of the references cited here contains numerous additional references to the various applications of (1.1). For example (1.1) describes the motion of a thin liquid film spreading under gravity, the flow in thin saturated regions in porous media and the percolation of a gas through a porous medium. A brief account of these particular applications and the original references can be found in Lacey, Ockendon and Tayler [19] which also includes references to other applications.

As far as the author is aware the majority of known "exact" solutions of (1.1) turn out to be similarity solutions even though originally they might have been derived say by a separation of variables technique or as a travelling wave solution. (See for example, Boyer [7] or Ames [1] (page 150) for a separation of variables derivation of the similarity source solution (2.6).) The known exact solutions of (1.1) are briefly summarized in the following section. If we examine similarity solutions of (1.1) then it is well known (see for example either Bluman and Cole [6] (page 295) or Hill [16] (page 140)) that there are three cases for the diffusivity $D(c)$ to consider, namely $D(c)$ arbitrary, $D(c)$ as a general power law $\alpha(c + \beta)^m$ where α, β and m denote constants and $D(c)$ as a power law but with index $m = -4/3$.

Here we shall principally be concerned with the general power law diffusivity and we take (1.1) in the form

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right), \tag{1.2}$$

noting that in all the solutions mentioned below we can change x and t to $(x - x_0)$ and $(t - t_0)$ respectively where x_0 and t_0 are arbitrary constants. Moreover with appropriate care we can change t to $-t$ and c to $c + \beta$ and we leave the details for such modifications to the reader. It is not difficult to show that (1.2) remains invariant under the one-parameter group

$$x_1 = e^{(\lambda+1)\varepsilon}x, \quad t_1 = e^{2\varepsilon}t, \quad c_1 = e^{2\lambda\varepsilon/m}c, \tag{1.3}$$

where λ is an arbitrary constant and accordingly (1.2) admits similarity solutions of the form

$$c(x, t) = x^{2\lambda/m(1+\lambda)}\phi(\xi), \quad \xi = \frac{x^{1/(1+\lambda)}}{t^{1/2}}, \tag{1.4}$$

for $\lambda \neq -1$ and where ϕ denotes a function of ξ which is determined by substitution of (1.4) into (1.2) and solving the resulting second-order ordinary differential equation. We remark that the integration procedure proposed here hinges on using the form $(1.4)_1$ rather than the form

$$c(x, t) = t^{\lambda/m}\psi(\xi), \tag{1.5}$$

which is commonly adopted in the literature, where $\psi(\xi) = \xi^{2\lambda/m}\phi(\xi)$. Moreover we have chosen the constant λ to parameterize the solution so that λ zero provides the similarity solution

$$c(x, t) = \phi(\xi), \quad \xi = \frac{x}{t^{1/2}}, \tag{1.6}$$

which applies to all nonlinear diffusivities $D(c)$.

In the following section we summarize the main known exact solutions of nonlinear diffusion. In the section thereafter we examine the second-order ordinary differential equation arising from (1.2) and (1.4) and propose a new integration procedure which generates values of λ for which the second-order differential equation admits a first integral. For one-dimensional diffusion with power law diffusivity, the two values of λ providing a first integral are

$$\lambda = \frac{-m}{(m+2)}, \quad \lambda = \frac{-m}{(m+1)}, \tag{1.7}$$

and curiously enough these are precisely the values of λ which apply for the point source solution (2.6) and the so-called "dipole-solution" (2.7) respectively. Moreover these solutions emerge from our analysis when the constant of integration in the first integral is set to zero. Our integration procedure is both elementary and quite general and in subsequent sections we show that a first integral can be deduced in a wide variety of situations, which include the exponential diffusivity, the power law diffusivity of index $-4/3$ for which a more general similarity solution applies, symmetrical diffusion in cylinders and spheres with power law diffusivity and one-dimensional diffusion but with an inhomogeneous power law diffusivity of the form

$$D(x, c) = c^m x^n. \tag{1.8}$$

We also demonstrate that if the second-order differential equation is known to admit a first integral then our procedure seems to provide a systematic process of determining such a first integral. For example Grundy [15] notes a special exact solution which emerges quite naturally from our approach. This is done in Section 4.

Finally in this section we note other approaches to similarity solutions of nonlinear diffusion. Grundy [15] provides an extensive phase plane analysis, Babu and Van Genuchten [2] investigate perturbation solutions while Shampine [26, 27] and Gilding and Peletier [13, 14] examine questions relating to existence and uniqueness. We also remark, as noted by Lacey, Ockendon and Tayler [19] that the potential use of the comparison theorem of Oleinik, Kalashnikov and Shzou Yui-Lin [21] provides further motivation for the construction of as many diverse similarity solutions as possible which gives additional justification to this study.

2. A brief survey of known solutions

The similarity solution (1.6) is of particular interest in soil science. From (1.1) and (1.6) we may readily deduce

$$(D(\phi)\phi')' + \frac{\xi}{2}\phi' = 0, \tag{2.1}$$

where here and throughout primes denote differentiation with respect of ξ. With boundary conditions $\phi(0) = 0$ and $\phi(\infty) = 1$, Philip [25] and Brutsaert [8] integrate (2.1) to obtain

$$D(\phi) = \frac{1}{2}\frac{d\xi}{d\phi}\int_\phi^1 \xi(\eta)\, d\eta, \tag{2.2}$$

and then for appropriate $\xi(\phi)$ calculate the diffusivity $D(\phi)$ by means of (2.2) so as to obtain an exact solution. Thus for example one such pair is

$$\xi(\phi) = \alpha\phi^\beta, \quad D(\phi) = \frac{\alpha^2\beta\phi^{\beta-1}}{2(\beta+1)}(1 - \phi^{\beta+1}), \tag{2.3}$$

and an extensive table of other similar examples can be found in [8]. Fujita [10, 11] obtains the exact general solutions of (2.1) in parametric form for power law diffusivities

$$D(c) = \alpha(c + \beta)^{-1}, \quad D(c) = \alpha(c + \beta)^{-2}. \tag{2.4}$$

Moreover Fujita [12] gives the general solution of (2.1) for

$$D(c) = \alpha(c^2 + \gamma c + \beta)^{-1}, \tag{2.5}$$

where α, β and γ denote arbitrary constants. The Fujita solutions although exact are complicated and need to be utilized with considerable care. The power law diffusivity of index -2 (namely $(2.4)_2$) relates to the work of Storm [29] and Hill and Hart [17] have utilized Fujita type solutions to obtain explicit representations for the Stefan problem in nonlinear heat condition. The exact solutions of Knight and Philip [18] also relate to the power law diffusivity of index -2. For the general power law diffusivity, Parlange and Braddock [22] and Parlange, Braddock and Chu [23] exploit transformations which give rise to first-order ordinary differential equations which are solved numerically. In the remainder of this section we catalogue known exact similarity solutions of (1.2) of the form (1.4) for which λ is non-zero.

For λ non-zero there are four known exact solutions each involving an arbitrary constant C. The first corresponds to diffusion from an instantaneous point source and appears to have been utilized by several authors (for example, Barenblatt [3], Zel'dovich and Kompaneets [31], Lonngren, Ames, Hirose and Thomas [20], Pattle [24], Boyer [7] and Tuck [30]). This solution occurs for $\lambda = -m/(m + 2)$ and is given by

$$c(x, t) = \frac{1}{x}\left(C\xi^{2m/(m+2)} - \frac{m\xi^2}{2(m + 2)}\right)^{1/m}, \tag{2.6}$$

for $\xi^{4/(m+2)} \leqslant 2(m + 2)C/m$ and zero otherwise where $\xi = x^{(m+2)/2}/t^{1/2}$. The second solution due to Barenblatt and Zel'dovich [5] and noted in both Lacey, Ockendon and Tayler [19] and Gilding and Peletier [13] occurs for $\lambda = -m/(m + 1)$ and is given by

$$c(x, t) = \frac{1}{x^2}\left(C\xi^{m(3+2m)/(m+1)^2} - \frac{m\xi^2}{2(m+2)}\right)^{1/m}, \tag{2.7}$$

for $\xi^{(m+2)/(m+1)^2} \leqslant 2(m + 2)C/m$ and zero otherwise where here $\xi = x^{(m+1)}/t^{1/2}$. We observe that as C tends to zero we obtain from both (2.6) and (2.7) the well known "waiting-time" solution discussed extensively by Lacey, Ockendon and Tayler [19], thus

$$c(x, t) = \left(\frac{-mx^2}{2(m + 2)t}\right)^{1/m}, \tag{2.8}$$

which can be deduced by the technique of separation of variables but is evidently also a similarity solution. The remaining two similarity solutions of (1.2) which involve an arbitrary constant C are the solution which is a function of x only, namely

$$c(x, t) = Cx^{1/(m+1)}, \tag{2.9}$$

which applies for $\lambda = m/(m + 2)$ and the travelling wave solution (see Barenblatt [4])

$$c(x, t) = (mC(Ct - x))^{1/m}, \tag{2.10}$$

for $x < Ct$ and zero otherwise and this applies for $\lambda = 1$.

Finally in this section we note that (1.2) admits solutions of the form

$$c(x, t) = f(x)/t^{1/m}, \tag{2.11}$$

which includes the waiting-time solution (2.8). From (1.2) and (2.11) we may readily deduce

$$(f^m f')' + \frac{f}{m} = 0, \tag{2.12}$$

where here primes denote differentiation with respect to x. Equation (2.12) can be integrated in a routine manner to give the first integral

$$f'^2 f^{2m} = C - \frac{2f^{m+2}}{m(m+2)}, \tag{2.13}$$

from which we may deduce

$$\int^{f(x)} \frac{\eta^m \, d\eta}{\left(C - \dfrac{2\eta^{m+2}}{m(m+2)}\right)^{1/2}} = \pm(x - x_0), \tag{2.14}$$

and we may readily verify that (2.8) emerges from equation (2.14) when the arbitrary constant C is zero.

3. Power law diffusivity

On substituting (1.4) into (1.2) we may readily deduce the second-order ordinary differential equation

$$\xi^2(\phi^m \phi'' + m\phi^{m-1}\phi'^2) + \lambda\left(\frac{4}{m} + 3\right)\xi\phi^m\phi'$$
$$+ \frac{2\lambda}{m}\left(\frac{2\lambda}{m} + \lambda - 1\right)\phi^{m+1} + \frac{(\lambda+1)^2}{2}\xi^3\phi' = 0. \tag{3.1}$$

Although we make no essential use of the following, we note that (3.1) is usually reduced to a first-order equation by making the substitution $\phi = \xi^{2/m}u(\xi)$ which generates an equation of the Euler type. Thus with

$$\eta = \log \xi, \quad p = \frac{du}{d\eta} = \xi \frac{du}{d\xi}, \tag{3.2}$$

we may deduce the first-order equation

$$u^m p \frac{dp}{du} + mu^{m-1}p^2 + (\lambda + 1)\left(\frac{4}{m} + 3\right) pu^m$$
$$+ (\lambda + 1)^2 \frac{2}{m}\left(\frac{2}{m} + 1\right) u^{m+1} + \frac{(\lambda + 1)^2}{2}\left(p + \frac{2u}{m}\right) = 0. \tag{3.3}$$

From this equation it is at least evident that the power law diffusivities with indices $m = -2$ and $m = -4/3$ play privileged roles, as is well known.

The integration procedure exploited here hinges on the observation that equation (3.1) consists of two types of terms, namely those arising from the spatial derivatives and that arising from the time differentiation in (1.2). Accordingly we divide (3.1) by ξ^3 so that the term arising from the time differentiation is ready for integration and we arrange the remainder in two steps, thus

$$-\frac{(\lambda + 1)^2}{2}\phi' = \left(\frac{\phi^m\phi'}{\xi}\right)' + \left(\frac{4\lambda}{m} + 3\lambda + 1\right)\frac{\phi^m\phi'}{\xi^2} + \frac{2\lambda}{m}\left(\frac{2\lambda}{m} + \lambda - 1\right)\frac{\phi^{m+1}}{\xi^3}$$
$$= \left[\frac{\phi^m\phi'}{\xi} + \left(\frac{4\lambda}{m} + 3\lambda + 1\right)\frac{\phi^{m+1}}{(m + 1)\xi^2}\right]'$$
$$+ \left[\frac{2\lambda}{m}\left(\frac{2\lambda}{m} + \lambda - 1\right) + \frac{2}{(m + 1)}\left(\frac{4\lambda}{m} + 3\lambda + 1\right)\right]\frac{\phi^{m+1}}{\xi^3}, \tag{3.4}$$

and clearly if the final term of this equation vanishes then (3.1) admits a first integral. This term vanishes provided λ is a root of the quadratic

$$(m + 1)(m + 2)\lambda^2 + m(2m + 3)\lambda + m^2 = 0, \tag{3.5}$$

which has roots given by (1.7). If $\lambda = -m/(m + 2)$ then from (3.4) we have the first integral

$$\frac{\phi^m\phi'}{\xi} - \frac{2\phi^{m+1}}{(m + 2)\xi^2} + \frac{2\phi}{(m + 2)^2} = C_1, \tag{3.6}$$

where C_1 is the integration constant. If this constant is zero then from (3.6) we have

$$\frac{\phi^{m-1}\phi'}{\xi} - \frac{2\phi^m}{(m + 2)\xi^2} = -\frac{2}{(m + 2)^2}, \tag{3.7}$$

which with the substituation $z = \phi^m$ integrates immediately to give the point source solution (2.6). On the other hand if $\lambda = -m/(m+1)$ then (3.4) gives the first integral

$$\frac{\phi^m \phi'}{\xi} - \frac{(2m+3)\phi^{m+1}}{(m+1)^2 \xi^2} + \frac{\phi}{2(m+1)^2} = C_1, \tag{3.8}$$

and if C_1 is zero we have

$$\frac{\phi^{m-1} \phi'}{\xi} - \frac{(2m+3)\phi^{m}}{(m+1)^2 \xi^2} = -\frac{1}{2(m+1)^2}, \tag{3.9}$$

which with the same substitution $z = \phi^m$ integrates immediately to give the dipole solution (2.7). Although the above procedure is completely elementary we now demonstrate that the routine is effective in producing a first integral in a variety of situations.

4. Exponential diffusivity and similarity solutions involving exponentials

In this section it is convenient to deal with the two special situations of the exponential diffusivity $D(c) = \alpha e^{\beta c}$ and the similarity solutions involving the similarity variable $xe^{-\alpha m t/2}$ where as usual α and β denote arbitrary constants. We first deal with the exponential diffusivity which arises from the power law diffusivity in the limit as m tends to infinity and in this limit we see from (1.7) that the values of λ for which the second-order equation admits a first integral both tend to minus one. As described in Hill [16] (problem 2, page 150) the full similarity solution appropriate to the general power law diffusivity $D(c) = \alpha(c + \beta^*)^m$ takes the form

$$(c + \beta^*)^m = [(1+\lambda)x + \kappa]^{2\lambda/(1+\lambda)} \phi^*(\omega^*), \tag{4.1}$$

where the similarity variable ω^* is defined by

$$\omega^* = \frac{[(1+\lambda)x + \kappa]^{1/(1+\lambda)}}{t^{1/2}}, \tag{4.2}$$

where λ has the same meaning as before and κ is an arbitrary constant. Now with

$$\beta^* = \frac{m}{\beta}, \quad \kappa = \frac{1}{\gamma}, \tag{4.3}$$

it is not difficult to show that the appropriate similarity solution arising from (4.1) and (4.2) in the limit as $m \to \infty$ and $\lambda \to -1$ is given by

$$e^{\beta c} = e^{-2\gamma x} \phi(\omega), \quad \omega = \frac{e^{\gamma x}}{t^{1/2}}, \tag{4.4}$$

and moreover we anticipate that the second-order ordinary differential equation for ϕ admits a first integral, as indeed it does. From the nonlinear diffusion equation (1.1) with exponential diffusivity $D(c) = \alpha e^{\beta c}$ we have from (4.4)

$$-\frac{\omega^3\phi'}{2\phi} = \frac{\alpha\gamma^2}{\beta}(\omega^2\phi'' - 3\omega\phi' + 4\phi), \tag{4.5}$$

where here primes of course denote differentiation with respect to ω. On dividing through by ω^3 we may readily deduce the first integral

$$\frac{\phi'}{\omega} - \frac{2\phi}{\omega^2} + \frac{\beta}{2\alpha\gamma^2}\log\phi = C_1, \tag{4.6}$$

as expected. Finally for completeness, for the exponential diffusivity we also note the solution

$$e^{\beta c} = \frac{(x - x_0)^2 + C}{2\alpha(t_0 - t)}, \tag{4.7}$$

where x_0, t_0 and C are all constants. This solution, which was originally noted in Smyth and Hill [28], can be readily deduced from the assumption $c(x, t) = f(x) + g(t)$ and represents the limiting solution which corresponds to (2.11) as m tends to infinity.

Another class of similarity solutions discussed by Gilding and Peletier [13, 14] and Grundy [15], applicable to the nonlinear diffusion equation with power law diffusivity, is given by

$$c(x, t) = x^{2/m}\phi(\xi), \quad \xi = xe^{-\alpha mt/2}, \tag{4.8}$$

where α is a constant. From (1.2) and (4.8) we have

$$-\frac{\alpha m}{2}\xi\phi' = \xi^2(\phi^m\phi'' + m\phi^{m-1}\phi'^2) + 4\left(1 + \frac{1}{m}\right)\xi\phi^m\phi' + \frac{2}{m}\left(1 + \frac{2}{m}\right)\phi^{m+1}, \tag{4.9}$$

which on dividing by ξ gives

$$-\frac{\alpha m}{2}\phi' = \left\{\xi\phi^m\phi' + \left(3 + \frac{4}{m}\right)\frac{\phi^{m+1}}{(m+1)}\right\}' + \frac{2}{m}\left(1 + \frac{2}{m}\right)\frac{\phi^{m+1}}{\xi}, \tag{4.10}$$

which can be integrated in the case $m = -2$ and we obtain the first integral

$$\xi\frac{\phi'}{\phi^2} - \frac{1}{\phi} - \alpha\phi = C_1. \tag{4.11}$$

This can be rearranged to provide the separable integral

$$\int^{\phi(\xi)} \frac{d\eta}{\eta(\alpha\eta^2 + C_1\eta + 1)} = \log \xi + C_2, \tag{4.12}$$

for which various explicit forms can be obtained depending on the constants α and C_1. This is precisely the exact solution noted by Grundy [15] in an appendix.

5. Power law diffusivity of index −4/3

As noted previously the nonlinear diffusion equation (1.1) with special power law diffusivity $D(c) = \alpha(c + \beta)^{-4/3}$ admits a wider class of similarity solutions. From either Bluman and Cole [6] (page 297) or Hill [16] (page 144) we can show that the more general similarity solution takes the form

$$c + \beta = \frac{\phi_1(\xi_1)}{[(x - x_1)^{1-\sigma}(x - x_2)^{1+\sigma}]^{3/2}}, \tag{5.1}$$

where the similarity variable ξ_1 is defined by

$$\xi_1 = \frac{1}{(t + \delta)^{1/2}} \left| \frac{x - x_1}{x - x_2} \right|^{\sigma}, \tag{5.2}$$

where x_1 and x_2 denote the roots of the quadratic equation,

$$\mu x^2 + (1 + \lambda)x + \kappa = 0, \tag{5.3}$$

assuming $(1 + \lambda)^2 > 4\mu\kappa$, that is

$$\begin{aligned} x_1 &= \{-(1 + \lambda) + [(1 + \lambda)^2 - 4\mu\kappa]^{1/2}\}/2\mu, \\ x_2 &= \{-(1 + \lambda) - [(1 + \lambda)^2 - 4\mu\kappa]^{1/2}\}/2\mu, \end{aligned} \tag{5.4}$$

and σ denotes $[(1 + \lambda)^2 - 4\mu\kappa]^{-1/2}$. We note that the functional form (5.1) and similarity variable (5.2) are obtained by solving the first-order partial differential equation

$$[\mu x^2 + (1 + \lambda)x + \kappa] \frac{\partial c}{\partial x} + 2(t + \delta) \frac{\partial c}{\partial t} = -\frac{3}{2}(c + \beta)(2\mu x + \lambda), \tag{5.5}$$

by the usual Lagrange characteristic method which is adequately described in either of the two references cited. We also note that here we have included translational constants κ, δ and β on x, t and c respectively. The constants δ and β are not essential but it is useful to retain κ non-zero. Finally we remark that with κ, δ and β all zero, the similarity solution (1.4) arises from the above in the limit as μ tends to zero, provided that we take appropriate care with the limiting process. In particular we need the results,

$$x_1 = -\frac{\kappa}{(1+\lambda)} + O(\mu), \quad x_2 = -\frac{(1+\lambda)}{\mu} + O(1),$$
$$\xi_1 \simeq \left(\frac{\mu}{(1+\lambda)}\right)^{1/(1+\lambda)} \xi, \quad \phi_1 \simeq \left(\frac{\mu}{(1+\lambda)}\right)^{-3(2+\lambda)/2(1+\lambda)} \phi, \tag{5.6}$$

where ξ and ϕ are understood to be that involved in (1.4) with $m = -4/3$.

After a long calculation we can deduce from (1.1) with $D(c) = \alpha(c+\beta)^{-4/3}$ and the above, the following second-order ordinary differential equation

$$\xi_1^2(\phi_1^{-4/3}\phi_1'' - \tfrac{4}{3}\phi_1^{-7/3}\phi_1'^2) + \tfrac{3}{4}\phi_1^{-1/3}[(1+\lambda)^2 - 4\mu\kappa - 1] + \frac{\mu^2}{2\alpha}\xi_1^3\phi_1' = 0. \tag{5.7}$$

Before proceeding to a first integral of (5.7) we observe for completeness that this equation can be reduced to a first-order equation by means of the substitution $\phi_1 = \xi_1^{-3/2}u_1(\xi_1)$ which again generates an equation of the Euler type, so that with

$$\eta_1 = \log \xi_1, \quad p_1 = \frac{du_1}{d\eta_1} = \xi_1\frac{du_1}{d\xi_1}, \tag{5.8}$$

we may deduce

$$p_1\frac{dp_1}{du_1} - \frac{4p_1^2}{3u_1} + \frac{3}{4}u_1[(1+\lambda)^2 - 4\mu\kappa] + \frac{\mu^2}{2\alpha}\left(p_1 - \frac{3u_1}{2}\right)u_1^{4/3} = 0. \tag{5.9}$$

In the limit μ tending to zero, this equation can be reconciled completely with (3.3) with $m = -4/3$ noting that from $(5.6)_{3,4}$ and the equations

$$u_1 = \xi_1^{3/2}\phi_1, \quad p_1 = \xi_1\frac{d}{d\xi_1}(\xi_1^{3/2}\phi_1), \tag{5.10}$$

we may deduce the approximate relations

$$u_1 \simeq \left(\frac{1+\lambda}{\mu}\right)^{3/2} u, \quad p_1 \simeq \left(\frac{1+\lambda}{\mu}\right)^{3/2} p, \tag{5.11}$$

valid for μ tending to zero where u and p are precisely as defined in Section 3 with $m = -4/3$. On using (5.11) in (5.9) we may confirm that in the limit as μ tends to zero we obtain precisely equation (3.3) with $m = -4/3$.

In order to deduce a first integral of (5.7) we follow the usual procedure and divide by ξ_1^3 to obtain

$$\left(\frac{\phi_1^{-4/3}\phi_1'}{\xi_1} - \frac{3\phi_1^{-1/3}}{\xi_1^2}\right)' + \frac{3\phi_1^{-1/3}}{4\xi_1^3}[(1+\lambda)^2 - 4\mu\kappa - 9] + \frac{\mu^2}{2\alpha}\phi_1' = 0, \tag{5.12}$$

which evidently can be integrated provided that the constants, λ, μ and κ are such that

$$(1 + \lambda)^2 - 4\mu\kappa = 9, \tag{5.13}$$

and the first integral in this case becomes

$$\frac{\phi_1^{-4/3}\phi_1'}{\xi_1} - \frac{3\phi_1^{-1/3}}{\xi_1^2} + \frac{\mu^2}{2\alpha}\phi_1 = C_1. \tag{5.14}$$

We have purposely retained the translational freedom in x in this section because the constraint (5.13) evidently involves the constant κ. If κ is zero then (5.13) immediately gives either $\lambda = 2$ or $\lambda = -4$. If the integration constant C_1 is zero then (5.14) gives

$$\frac{\phi_1^{-7/3}\phi_1'}{\xi_1} - \frac{3\phi_1^{-4/3}}{\xi_1^2} = -\frac{\mu^2}{2\alpha}, \tag{5.15}$$

which with the substitution $z_1 = \phi_1^{-4/3}$ can be readily integrated to yield

$$\phi_1(\xi_1) = \left(C\xi_1^{-4} + \frac{\mu^2\xi_1^2}{9\alpha}\right)^{-3/4}, \tag{5.16}$$

which is a new exact solution valid provided that λ, μ and κ are such that (5.13) holds. If this is the case then $\sigma = 1/3$ and (5.1) and (5.2) give

$$c + \beta = \frac{\phi_1(\xi_1)}{(x - x_1)(x - x_2)^2}, \quad \xi_1 = \frac{1}{(t + \delta)^{1/2}}\left|\frac{x - x_1}{x - x_2}\right|^{1/3}, \tag{5.17}$$

where x_1 and x_2 are given by

$$x_1 = (2 - \lambda)/2\mu, \quad x_2 = -(4 + \lambda)/2\mu, \tag{5.18}$$

and we observe that if the constant κ is zero, then the two possible values of λ (namely $\lambda = 2$ or $\lambda = 4$) are such that either x_1 or x_2 is zero.

Finally in this section we compare this new solution with the point source and dipole solutions (2.6) and (2.7) respectively. For comparison purposes we take. β, δ and κ to be zero so that $\lambda = 2$ or $\lambda = -4$. For $m = -4/3$ we have $\lambda = -m/(m + 2) = 2$ and the point source solution (2.6) becomes

$$c(x, t) = \frac{(C\xi^{-4} + \xi^2)^{-3/4}}{x} = \left(\frac{t}{x^2 + Ct^3}\right)^{3/4}, \quad \xi = \frac{x^{1/3}}{t^{1/2}}, \tag{5.19}$$

while if $\lambda = 2$ the new solution (5.17) gives

$$c(x, t) = \frac{(C\xi_1^{-4} + B\xi_1^2)^{-3/4}}{x(x + a)^2} = \frac{1}{(x + a)^{3/2}}\left(\frac{t}{Bx^2 + Ct^3(x + a)^2}\right)^{3/4}, \quad \xi_1 = \frac{x^{1/3}}{t^{1/2}(x + a)^{1/3}}, \tag{5.20}$$

where $a = 3/\mu$ and $B = \mu^2/9\alpha$. On the other hand for $m = -4/3$ we have $\lambda = -m/(m+1) = -4$ and the dipole solution (2.7) becomes

$$c(x, t) = \frac{(C\xi^{-4} + \xi^2)^{-3/4}}{x^2} = \frac{1}{x^{3/2}}\left(\frac{t}{1 + Ct^3x^2}\right)^{3/4}, \quad \xi = \frac{1}{t^{1/2}x^{1/3}}, \tag{5.21}$$

while if $\lambda = -4$ the new solution (5.17) gives

$$c(x, t) = \frac{(C\xi_1^{-4} + B\xi_1^2)^{-3/4}}{x^2(x - a)} = \frac{1}{x^{3/2}}\left(\frac{t}{B(x - a)^2 + Ct^3x^2}\right)^{3/4}, \quad \xi_1 = \frac{(x - a)^{1/3}}{t^{1/2}x^{1/3}}, \tag{5.22}$$

where a and B are precisely as previously defined. We observe that (5.22) is exactly (5.20) with a translation on x, that is, if in (5.20) we translate x to $x - a$ this gives precisely (5.22). Further on noting that (5.19) and (5.21) apply to (1 2) which has α set to unity, we observe that with this value of α, $B = a^{-2}$ and (5.19) and (5.21) emerge from the new solutions (5.20) and (5.22) respectively in the limit a tending to infinity, that is μ tending to zero.

6. First integrals for a general equation

In this section it is convenient to deal with the general equation

$$\frac{\partial c}{\partial t} = x^l \frac{\partial}{\partial x}\left\{c^m x^n \frac{\partial c}{\partial x}\right\}, \tag{6.1}$$

for arbitrary l, m and n, from which the two important special cases $l = -n$ and $l = 0$ detailed below can be readily deduced. It is not difficult to establish that (6.1) admits similarity solutions of the form

$$c(x, t) = x^{A/m}\phi(\xi), \quad \xi = \frac{x^{1/(1+\lambda)}}{t^{1/2}}, \tag{6.2}$$

where the constant A is defined by

$$A = \frac{2\lambda}{(1 + \lambda)} - (n + l). \tag{6.3}$$

On substituting (6.2) into (6.1) the resulting second-order ordinary differential equation can be rearranged to give

$$\begin{aligned} &\left\{\frac{\phi^m\phi'}{\xi} + \left(\frac{1+\lambda}{m+1}\right)\left(\frac{2A}{m} + 2A + \frac{2}{(1+\lambda)} + n - 1\right)\frac{\phi^{m+1}}{\xi^2} + \frac{(1+\lambda)^2}{2}\phi\right\}' \\ &\quad + \left\{\frac{A}{m}\left(\frac{A}{m} + A + n - 1\right)(1+\lambda)^2 + 2\left(\frac{1+\lambda}{m+1}\right)\right. \\ &\quad \left. \times \left(\frac{2A}{m} + 2A + \frac{2}{(1+\lambda)} + n - 1\right)\right\}\frac{\phi^{m+1}}{\xi^3} = 0, \end{aligned} \tag{6.4}$$

which clearly admits a first integral provided the final term vanishes and the condition for the vanishing of this term is a quadratic equation in A with the two roots,

$$A = \frac{-m}{(m+1)}\left\{\frac{2}{(1+\lambda)} + n - 1\right\}, \quad A = \frac{-2m}{(m+1)(1+\lambda)}. \tag{6.5}$$

These equations can be rearranged to give the following values of λ for which a first integral can be readily obtained, namely

$$\lambda = \left(\frac{l(m+1) + n - m}{m + 2 - l(m+1) - n}\right), \quad \lambda = \frac{2m - (l+n)(m+1)}{(m+1)(n+l-2)}, \tag{6.6}$$

and the first integrals corresponding to these values are, respectively,

$$\begin{aligned} &\frac{\phi^m\phi'}{\xi} - \left(\frac{1+\lambda}{m+1}\right)\left(n - 1 + \frac{2}{(1+\lambda)}\right)\frac{\phi^{m+1}}{\xi^2} + \frac{(1+\lambda)^2}{2}\phi = C_1, \\ &\frac{\phi^m\phi'}{\xi} + \left(\frac{1+\lambda}{m+1}\right)\left(n - 1 - \frac{2}{(1+\lambda)}\right)\frac{\phi^{m+1}}{\xi^2} + \frac{(1+\lambda)^2}{2}\phi = C_1. \end{aligned} \tag{6.7}$$

The first important special case of the above corresponds to symmetrical diffusion in cylinders and spheres and arises from the case $l = -n$ and $n = 1$ for cylinders while $n = 2$ for spheres. In this situation we obtain from (6.6) the following two values of λ,

$$\lambda = \frac{-m(n+1)}{(m+2+mn)}, \quad \lambda = \frac{-m}{(m+1)}, \tag{6.8}$$

with corresponding first integrals obtained from (6.7)

$$\begin{aligned} &\frac{\phi^m\phi'}{\xi} - \frac{2(n+1)}{(m+2+mn)}\frac{\phi^{m+1}}{\xi^2} + \frac{2\phi}{(m+2+mn)^2} = C_1, \\ &\frac{\phi^m\phi'}{\xi} - \frac{(2m+3-n)}{(m+1)^2}\frac{\phi^{m+1}}{\xi^2} + \frac{\phi}{2(m+1)^2} = C_1, \end{aligned} \tag{6.9}$$

which incidentally coincide with (3.6) and (3.8) respectively when n is zero. If we take the constants of integration to be zero then we obtain from (6.9) the appropriate multi-dimensional versions of the point source and dipole solutions (see for example, Barenblatt [3] and Barenblatt and Zel'dovich [5]). We observe, curiously enough that for the cylinder ($n = 1$) the two values of λ and the two integrals coincide.

The second important special case of the above corresponds to one-dimensional diffusion but with an inhomogeneous power law diffusivity of the form (1.8) and arises from the case $l = 0$. From (6.6) we have the following two values of λ,

$$\lambda = -\frac{(m-n)}{(m-n+2)}, \quad \lambda = \frac{2m - n(m+1)}{(m+1)(n-2)}, \tag{6.10}$$

while from (6.7) we may deduce the corresponding first integrals

$$\begin{aligned} &\frac{\phi^m \phi'}{\xi} - \frac{2}{(m-n+2)}\frac{\phi^{m+1}}{\xi^2} + \frac{2\phi}{(m-n+2)^2} = C_1, \\ &\frac{\phi^m \phi'}{\xi} - \frac{2[m(n-2) + 2n - 3]\phi^{m+1}}{(m+1)^2(n-2)\xi^2} + \frac{2\phi}{(m+1)^2(n-2)^2} = C_1. \end{aligned} \tag{6.11}$$

7. Conclusion

For a wide variety of situations involving similarity solutions of the nonlinear diffusion equation (1.1) we have proposed a simple procedure to deduce first integrals of the governing second-order ordinary differential equation and as far as the author is aware these results have not been given previously. The major known exact solutions (namely (2.6) and (2.7)) arise from our first integral when the constant of integration is taken to be zero and other special exact solutions emerge naturally from our procedure. A new exact solution for the power law diffusivity of index $-4/3$ is deduced in Section 5. The derived first integrals give rise to a large number of first-order ordinary differential equations, which although not immediately integrable by standard devices, nevertheless deserve consideration either for exact integrals in special cases or numerical integration for particular problems and these areas will form the basis of a future study.

Acknowledgement

Part of this work was undertaken while the author was a Visiting Scholar at the James Cook University of North Queensland. The hospitality of Brian Richards and Roger Hosking is gratefully acknowledged.

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